

The Physics of Stars
Assessed Coursework Assignment

URN: 1153196

1. Introduction

The star of interest for this report is UV Leonis B. It is a main sequence star on the Hertzsprung-Russell diagram and is one of a binary pair. Some of its main attributes are shown in figure 1 below.

Position		Spectrum	Mass (Solar masses)	Radius (Solar radii)	Abs. Bol. Mag.
Ra	Deg				
10 ^{hr} 33 ^{min}	+14°47'	G2	0.9	0.99	+4.7

Figure 1. Table showing main attributes for UV Leonis B.

This star is classified as a G2 with its colour index (B-V) being 0.64. The classification is made on a scale which uses the strength of the star's hydrogen absorption lines (Balmer series) and its effective surface temperature (5780K in this case). A G2 type star will produce weak Balmer lines, but stronger Calcium lines and a Planck wavelength peak in the yellow region of the electromagnetic spectrum.

UV Leonis B is in many ways similar to the Sun, in terms of radius, mass and classification, so this will give a useful indication of some of the starting points and fitting parameters when it comes to modelling the star.

This report aims to investigate some of the fundamental properties of the assigned star, by first carrying out some basic structure calculations and estimates of the main parameters such as temperature, pressure and luminosity. Two different structure models will then be employed to further investigate the star, using the simple estimates to validate the model calculations. Finally, the possible end points of UV Leonis B will be discussed.

2. Estimates of important parameters

It is important to estimate the fundamental parameters of the star in order to gain some understanding of its structure and composition.

2.1 Luminosity

$$M_{\text{bol}} = -2.5 \log_{10} \frac{L}{L_0} + 4.72 \quad [\text{Equation 1}]$$

where: M_{bol} = Absolute bolometric magnitude = +4.7
 L = Luminosity
 L_0 = Luminosity of Sun = $3.83 \times 10^{26} \text{W}$

$$\Rightarrow L = 3.90 \times 10^{26} \text{W}$$

2.2 Temperature

The effective surface temperature can be calculated from the luminosity:

$$L = 4\pi r^2 \sigma T^4 \quad [\text{Equation 2}]$$

where: T = Effective surface temperature
 r = Radius = 0.99 solar radii = $6.89 \times 10^8 \text{m}$
 σ = Stefan Boltzmann constant = $5.671 \times 10^{-8} \text{Wm}^{-2}\text{K}^{-2}$

$$\Rightarrow T = 5827 \text{K}$$

The mean temperature is then:

$$\langle T \rangle = \frac{GM\bar{m}}{6k_B R} \quad [\text{Equation 3}]$$

where: G = Gravitational constant = $6.7 \times 10^{-11} \text{m}^3 \text{kg}^{-1} \text{s}^{-2}$
 M = Mass of star = 0.9 solar masses = $1.79 \times 10^{30} \text{kg}$
 \bar{m} = Average particle mass = $2m_H / (1 + 3X_1 + 1/2X_4)$
 X_1 = Proportion of Hydrogen
 X_4 = Proportion of Helium
 m_H = Mass of Hydrogen = $1.673 \times 10^{-27} \text{kg}$
 k_B = Boltzmann constant = $1.381 \times 10^{-22} \text{JK}^{-1}$
 R = Radius of star = 0.99 solar radii = $6.89 \times 10^8 \text{m}$

$$\Rightarrow \langle T \rangle = 1.76 \times 10^6 \text{K} \quad (\text{Assuming } 100\% \text{ H})$$

$$\Rightarrow \langle T \rangle = 3.12 \times 10^6 \text{K} \quad (\text{Assuming } 70\% \text{ H}:30\% \text{ He})$$

As these values are taken over the whole depth of the star, the calculated average will be significantly higher than that of the surface value due to the extreme temperature of the core.

The calculation allows assumptions to be made about the composition of the star. For the purpose of this report the initial calculations have assumed the star to be composed of 100% ionised Hydrogen, but for a more realistic picture, the structure models were also run using the same ratio of Hydrogen:Helium as is found in the Sun (70%H:30%He).

The central temperature of the star can be estimated using:

$$T_c = \frac{GM\bar{m}}{2k_B R} \quad \text{[Equation 4]}$$

$$\begin{aligned} \Rightarrow T_c &= 5.27 \times 10^6 \text{K} \quad (100\% \text{ H}) \\ \Rightarrow T_c &= 9.37 \times 10^6 \text{K} \quad (70\% \text{ H}) \end{aligned}$$

2.3 Pressure

A lower boundary can be estimated for the central pressure (P_c) by using the fact that a point r from the centre of the star will always be less than the actual radius, R . The actual value for P_c will therefore always be greater than what is calculated, and will decrease radially outwards, tending to zero at the surface.

$$P_c > \frac{GM^2}{8\pi R^4} \quad \text{[Equation 5]}$$

$$\Rightarrow P_c > 3.79 \times 10^{13} \text{Pa}$$

By making the assumption that the star has a constant density throughout and is in hydrostatic equilibrium, the central pressure then becomes:

$$P_c = \frac{3GM^2}{8\pi R^4} \quad \text{[Equation 6]}$$

$$\Rightarrow P_c = 1.14 \times 10^{14} \text{Pa}$$

The Virial Theorem is another method of calculating pressure. It is an expression for the average pressure needed to support a self-gravitating system:

$$\langle \mathbf{P} \rangle \mathbf{V} = \frac{-\mathbf{E}_G}{3} \quad \text{[Equation 7]}$$

where: $V = \text{Volume of star} = 4/3\pi R^3$
 $E_G = \text{Gravitational energy}$

$$E_G = \frac{-GM^2}{2R} \quad \text{[Equation 8]}$$

$$\begin{aligned} \Rightarrow E_G &= -1.56 \times 10^{41} \text{J} \\ \Rightarrow \langle \mathbf{P} \rangle &= 5.5 \times 10^{14} \text{Pa} \end{aligned}$$

2.4 Non-relativistic gas pressure

During its main sequence lifetime, UV Leonis B will operate in a non-relativistic non-degenerate scheme, similar to the Sun. The gas pressure can therefore be expressed as:

$$\langle P \rangle V = \frac{2E_K}{3} \quad \text{[Equation 9]}$$

where: E_K = Kinetic energy of gas

Comparing this to the Virial theorem, an expression for kinetic energy is obtained:

$$E_K = -1/2 E_G \quad \text{[Equation 10]}$$

$$\Rightarrow E_K = 7.8 \times 10^{40} \text{J}$$

The total energy of the non-relativistic gas system, assuming no internal degrees of freedom (i.e. ignoring the internal mechanisms of storing energy) is then:

$$E_{TOT} = E_K + E_G \quad \text{[Equation 11]}$$

$$\Rightarrow E_{TOT} = -7.8 \times 10^{40} \text{J}$$

The large negative result means that the gas is tightly bound. The tighter it is bound, the hotter the star will be. If gravitational energy is lost, half of it must be used to heat the gas, and the other half is radiated away.

2.5 Radiation Pressure

The expression for the radiation pressure stems from Stefan's law relating luminosity and effective surface temperature:

$$P_{rad} = 1/3 a T^4 \quad \text{[Equation 12]}$$

Where: a = radiation density constant = $7.57 \times 10^{-16} \text{JK}^{-4}$

$$\Rightarrow P_{rad} = 0.29 \text{Pa}$$

2.4 Hydrogen burning

For hydrogen burning to occur, and energy to be produced, two protons have to first overcome the coulomb barrier via quantum tunnelling. In order to produce a stable nucleus the protons must fuse together, and some of them will decay into neutrons. This is known as the proton-proton chain and a number of reaction pathways exist for doing this.

Another method of energy production is via the CNO cycle. However, this process only dominates in stars with higher temperatures and mass, so the p-p chain will be the dominant mechanism in UV Leonis B.

2.5 Energy production

A simplified calculation to find the energy production from the p-p chain in a star similar to the Sun is found to be:

$$\varepsilon_{pp} = 9.5 \times 10^{-37} X_1^2 \rho^2 T^4 \quad [\text{Equation 13}]$$

$$\Rightarrow \varepsilon_{pp} = 297.6 \text{ Wm}^{-3}$$

This value is estimated at the centre of the star, assuming 70% Hydrogen.

2.6 Density

Assuming a constant density model, a value can be calculated from the mass and radius of the star, and assuming it to be spherical:

$$\langle \rho \rangle = \frac{3M}{4\pi R^3} \quad [\text{Equation 14}]$$

$$\Rightarrow \langle \rho \rangle = 1.31 \times 10^3 \text{ kgm}^{-3}$$

An estimate for central density can also be calculated by equating two expressions which have similar r dependence:

$$a^2 \rho_c^3 \approx \frac{16\pi\sigma}{10^{-39} X_1 \kappa} \quad [\text{Equation 15}]$$

where: a = radiation density constant
 σ = Stefan Boltzmann constant
 $\kappa = 0.02(1 + X_1)$

$$M \approx \frac{4\pi\rho_c a^3 \sqrt{6}}{3} \quad [\text{Equation 16}]$$

$$\Rightarrow \rho_c = 2.88 \times 10^5 \text{ kgm}^{-3}$$

This value expressed as a fraction of the mean solar density is a good starting estimate for the central density in the structure models.

3. Structure Calculations

This report uses two structure models, Polytrope and Clayton, created on an Excel spreadsheet in order to calculate the attributes of UV Leonis B. They work by applying a number of fitting parameters in order to satisfy certain boundary conditions.

3.1 Polytrope Model

This model assumes a simple power law relationship between P and ρ and consequently a dependence of T on ρ . It uses the Euler integration method to calculate pressure, temperature and mass from a series of equations and initial estimates of the fitting parameters. The equations used are:

$$\text{Pressure:} \quad P' = K' \rho'^{(n+1/n)} \quad [\text{Equation 17}]$$

$$\text{Pressure gradient:} \quad \frac{dP'}{dr'} = \frac{-2m'\rho'}{r'^2} \quad [\text{Equation 18}]$$

$$\text{Mass gradient:} \quad \frac{dM'}{dr'} = 3r'^2 \rho' \quad [\text{Equation 19}]$$

$$\text{Temperature:} \quad T' = \frac{m'P'}{m'_H \rho'} \quad [\text{Equation 20}]$$

$$\text{Euler equation:} \quad y_{i+1} = y_i + \Delta x \left. \frac{dy}{dx} \right|_{x=i} \quad [\text{Equation 21}]$$

These equations represent the quantities as a fraction of typical solar values, so all results are dimensionless, but can be converted back into their absolute values by multiplying through by the relevant solar attributes.

The values were calculated in increments of $0.01r$ (or Solar radii) from zero (i.e. the centre of the star), to 0.9 , which is the radius of this star. Three fitting parameters were then used in order to calculate the first values, before the Euler method continued the calculations through the steps.

$$\begin{aligned} \text{Fitting Parameters:} \quad n &= \text{Index of Polytrope model} \\ \rho_c &= \text{Central density} \\ K &= \text{A constant} \end{aligned}$$

These quantities are unknowns and need to be estimated in order to fulfill the two constraints:

$$\begin{aligned} \text{Constraints:} \quad P &\rightarrow 0 \text{ as } r \rightarrow R \\ m &\rightarrow M \text{ as } r \rightarrow R \end{aligned}$$

A good starting point for the values of the fitting parameters was to use the numbers quoted for a typical solar model. A first guess for n can be made by knowing some typical values which work for other stars. In the solar model for an adiabatic gas n is usually in the region of 1.5, while for a star with high temperature and density (*i.e.* ultra-relativistic and degenerate gases) this is set in the region of 3. By initially setting n to one of these numbers, the two remaining parameters can be altered to get a good fit which obeys the boundary conditions. A new value for n can then be chosen and the same procedure followed to get a new set of results. More physics can be applied, such as the calculations for energy and luminosity with aim to identifying which value of n best satisfies the estimated values.

However, the Polytrope model has limitations that a numerical integration is required and there are more fitting parameters than constraints.

3.2 Clayton Model

This model uses assumptions of the way pressure varies with r , and relates the two fitting parameters, so as one varies, the other changes with it. This ensures that the mass constraint is easily fulfilled.

$$\text{Pressure gradient: } \frac{dP'}{dr'} = -2\rho_c'^2 r' \exp\left[\frac{-r'^2}{a'^2}\right] \quad [\text{Equation 22}]$$

$$\text{Pressure: } P' = \rho_c'^2 a'^2 \left[\exp\left[\frac{-r'^2}{a'^2}\right] - \left[\frac{-R'^2}{a'^2}\right] \right] \quad [\text{Equation 23}]$$

$$\text{Mass: } m' = a'^3 \rho_c' \phi \quad [\text{Equation 24}]$$

$$\text{Density: } \rho' = \frac{\rho_c' r'^3}{\phi a'^3} \exp\left[\frac{-r'^2}{a'^2}\right] \quad [\text{Equation 25}]$$

$$\text{Temperature: } T' = \frac{\bar{m}P'}{m_H \rho'} \quad [\text{Equation 26}]$$

$$\text{where: } a' = \left[\frac{M'}{\sqrt{6}\rho_c'} \right]^{1/3} \quad [\text{Equation 27}]$$

$$\phi = \sqrt{6 - 3(x^4 + 2x^2 + 2)\exp(-x^2)} \quad [\text{Equation 28}]$$

$$x = r/a \quad [\text{Equation 29}]$$

These equations are also dimensionless, representing fractions of solar values.

Fitting Parameters: $a = A$ constant (height and length scale)
 $\rho_c =$ Central density

4. Results

4.1 Polytrope

A good starting point was to use the values of fitting parameters from the solar model. UV Leonis B has a similar mass and radius of the Sun, so by using these parameters, the same values for central pressure and temperature were obtained, and similar trends were followed. Some small adjustments allowed a more accurate picture of the star to be reached. By setting the Polytrope index to 3, another set of values were obtained which also obeyed the constraints. Figure 2 shows a selection of the values used by the model in order to produce a set of results which fulfill the boundary conditions.

Polytrope Index, n	\bar{m} / m_H	K'	P_c'
1.5	0.5 (100% H)	0.36	4
1.5	0.615 (70% H)	0.326	4.65
3	0.615	0.51	4

Figure 2. Table of fitting parameters examined.

Figure 3 shows the results using the fitting parameters from the standard solar model, assuming a composition of 100% Hydrogen. The trends for the mass, pressure and pressure gradient are as expected, with $M=0.94$ solar masses, although this is a little high, and pressure tending to zero at the surface. The value of central pressure is found to be 3.63, which if converted into absolute units gives $4.8 \times 10^{14} \text{Pa}$. This is calculated by multiplying by the mean solar pressure, quoted as $1.33 \times 10^{14} \text{Pa}$.

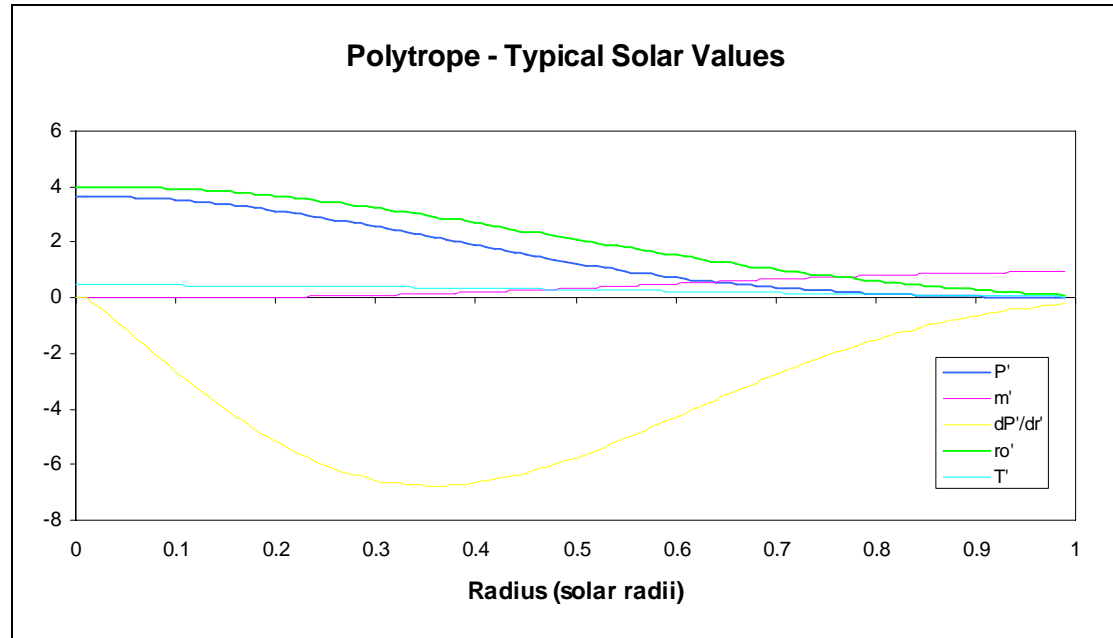


Figure 3. Model of UV Leonis B using fitting parameters from standard solar model.

By now assuming a composition ratio of 70% Hydrogen:30% Helium and making adjustments to K' and ρ_c' the model fits the constraints better (figure 4). The final mass reaches the required 0.9 solar masses, and central pressure is $5.6 \times 10^{14} \text{Pa}$. This value satisfies the estimates made earlier that the lower boundary, $P_c > 1.14 \times 10^{14} \text{Pa}$. The central temperature is also found to be $6.4 \times 10^6 \text{K}$, which is a little lower than the

estimated value of $9.3 \times 10^6 \text{K}$, but is in the right region. If the temperature is averaged over all the steps, a value of $3.83 \times 10^6 \text{K}$ is obtained. This compares very well with the result of $\langle T \rangle = 3.12 \times 10^6 \text{K}$ which was calculated earlier. Figure 4 shows all parameters normalised to their maximum values. This allows the trends to be visualized more easily.

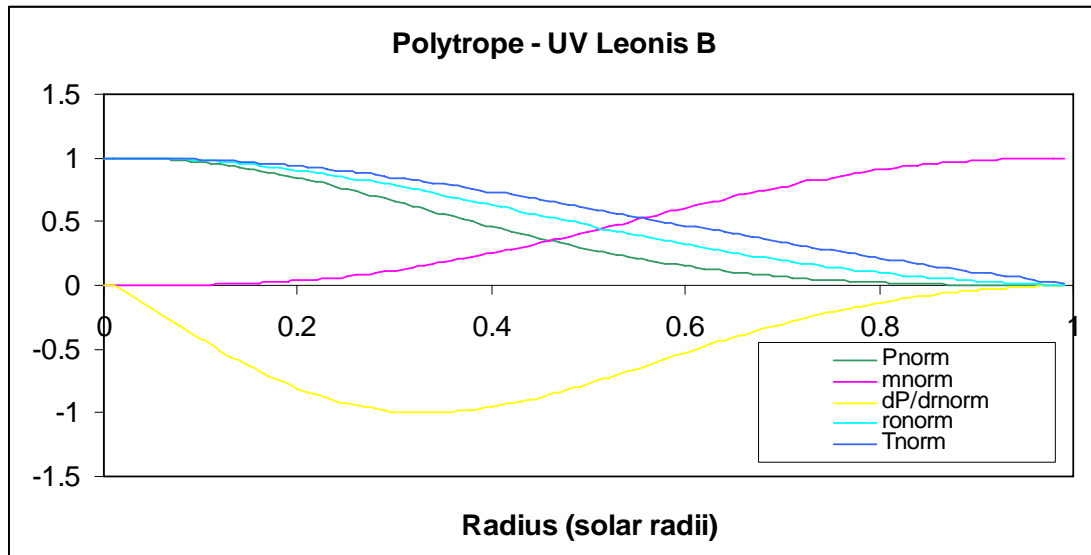


Figure 4. Making small adjustments to the standard solar values to obtain a better fit.

Now changing the Polytrope index to 3 (figure 5), a new set of parameters can be chosen to obtain a model that works. This time, the mass reaches 0.91 solar masses with central pressure and temperature being $4.30 \times 10^{14} \text{Pa}$ and $5.77 \times 10^6 \text{K}$ respectively. The mean temperature from this model is found to be $4.38 \times 10^6 \text{K}$. Again these values show a reasonably good agreement with the original calculations, although the temperature does not appear to be represented exactly as the original calculations would suggest.

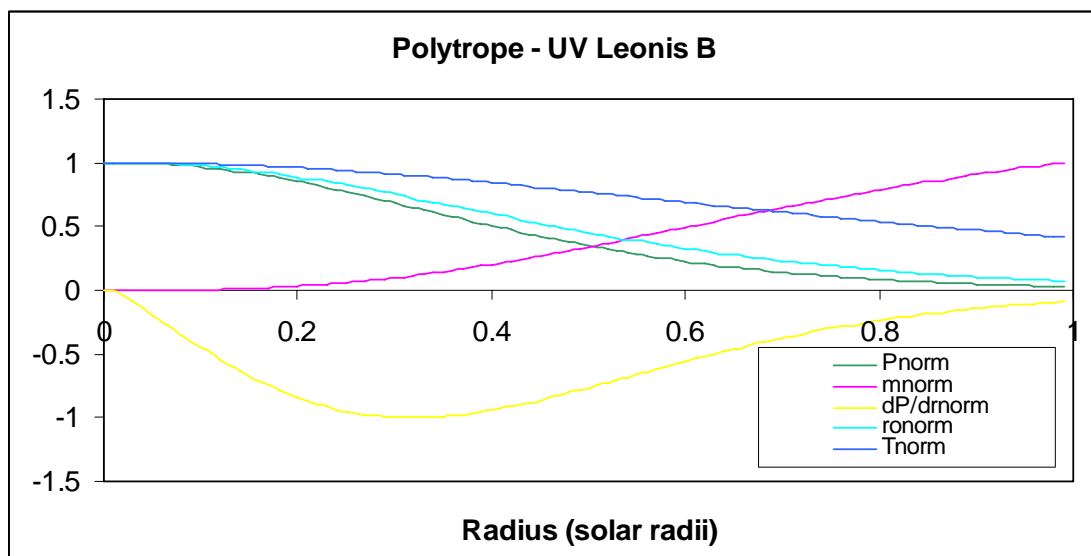


Figure 5. Normalised results setting the Polytrope Index to 3

The only way to narrow down which set of parameters are correct is to add more physics. This can be done by including calculations for energy production in the star and luminosity. The model uses equation 13 expressed in terms of solar units as a way of calculating the luminosity through the star. The equations input into the spreadsheet become:

$$\text{Energy production: } \epsilon_{pp}' = \rho'^2 T'^4 \quad [\text{Equation 30}]$$

$$\text{Luminosity gradient: } \frac{dL'}{dr'} = r'^2 \epsilon_{pp}' \quad [\text{Equation 31}]$$

$$\text{Luminosity: } L' = L_0 + \left(\frac{dL'}{dr'} \times \Delta r \right) \quad [\text{Equation 32}]$$

Using a Polytrope index of 1.5, and the fitting parameters shown in figure 2 produces a good result for the two attributes. Figure 6 shows that the energy produced via the proton-proton chain mainly occurs in the centre of the star where the temperatures are high enough to sustain thermonuclear reactions. The reaction rate then drops as radius increases and flattens out to a constant value at about 0.6 solar radii.

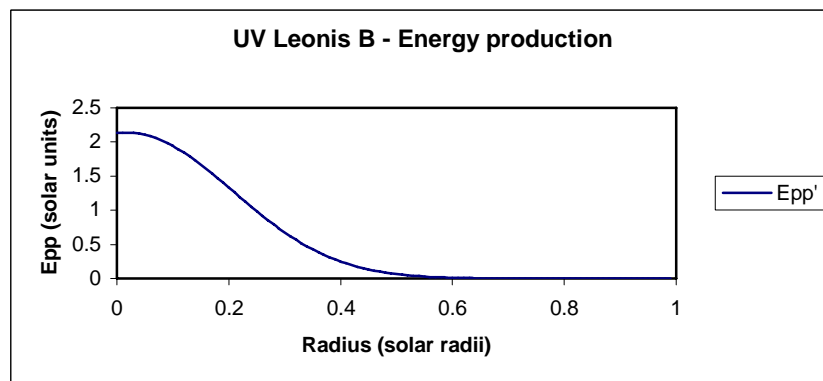


Figure 6. Energy production using a Polytrope index of 1.5

The luminosity via this method also produces a graph with the expected shape, and yields a value of 0.02 at the surface. Converting to absolute units gives $7.70 \times 10^{24} \text{W}$, which is a little lower than what is expected, but within two orders of magnitude. Figure 7 shows this result.

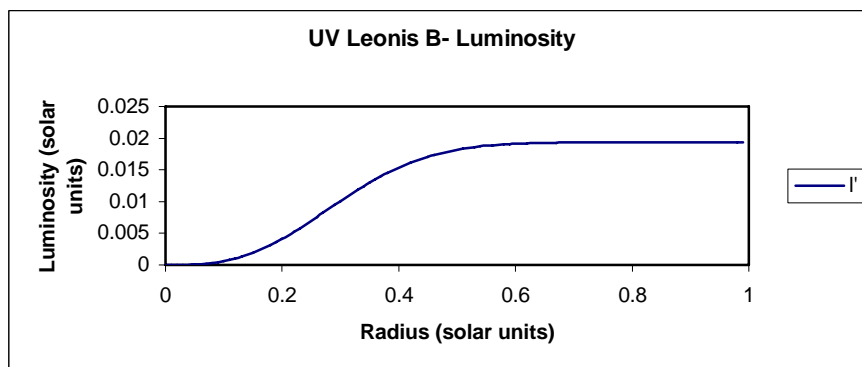


Figure 7. Luminosity from the Polytrope model

With the fitting parameters set to their best values when $n=3$, a new set of energy and luminosity curves can be obtained. Figure 8 shows these attributes as a function of radius, with results normalised to their maximum values. In this case the luminosity at the surface is $5.0 \times 10^{24} \text{W}$. This is lower than the value calculated using the previous fitting parameters, suggesting that a better estimate for the Polytrope index is closer to 1.5, similar to the Sun.

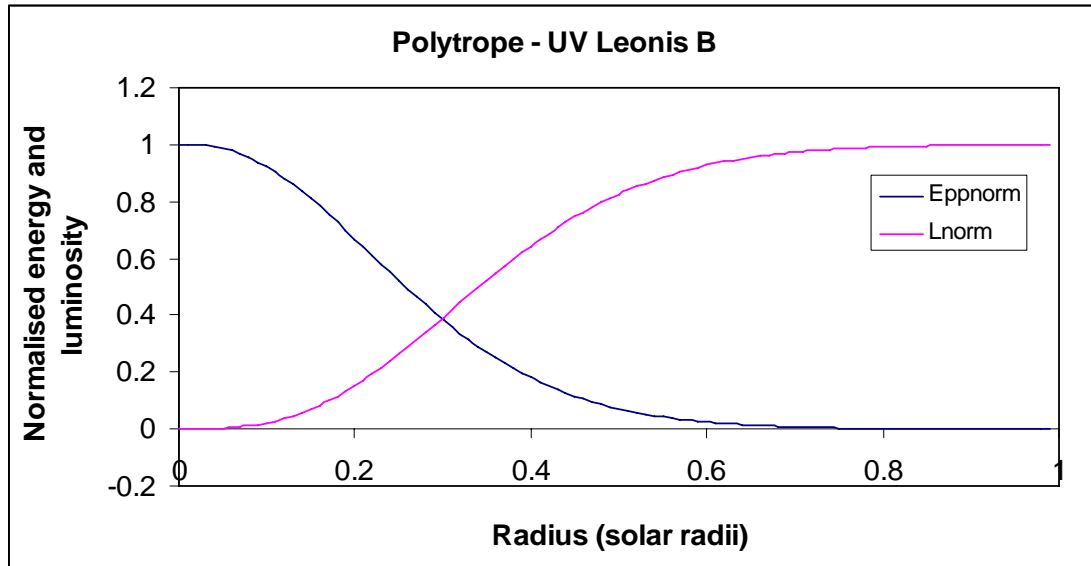


Figure 8. Normalised energy and luminosity from the Polytrope model

4.2 Clayton

By using this model it may be possible to resolve some of the limitations with temperature that were found in the Polytrope model. In this case the fitting parameter, a , varies with ρ_c , so adjusting this value allows the mass and pressure constraints to be satisfied more easily. The value of central density that worked best in the model was 25, or $3.4 \times 10^4 \text{kgm}^{-3}$ in absolute units. This yielded values of $P_c = 4.9 \times 10^{15} \text{Pa}$, $\langle P \rangle = 9.11 \times 10^{14} \text{Pa}$, $T_c = 1.07 \times 10^7 \text{K}$, and $\langle T \rangle = 4.18 \times 10^6 \text{K}$. Energy and luminosity are also well represented through the star, producing a surface value of $L = 2.52 \times 10^{26} \text{W}$, which is a much closer agreement to the estimated value from equation 1.

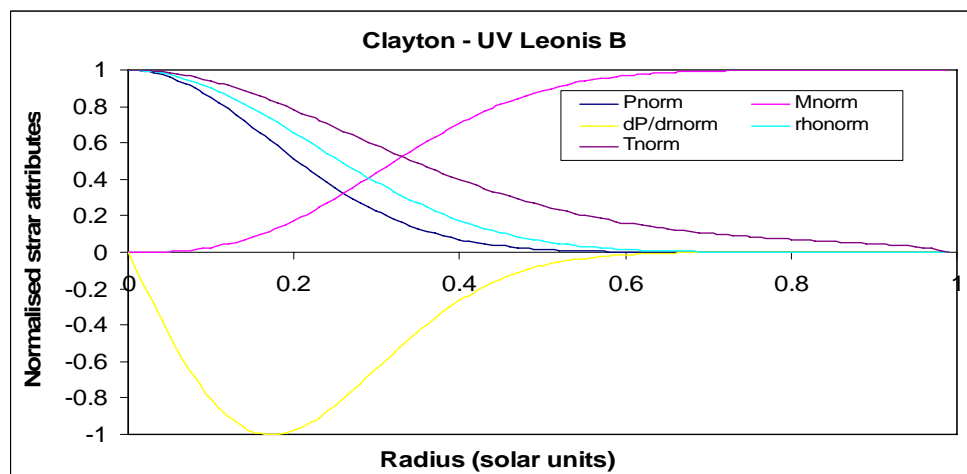


Figure 9. Normalised star attributes from the Clayton model.

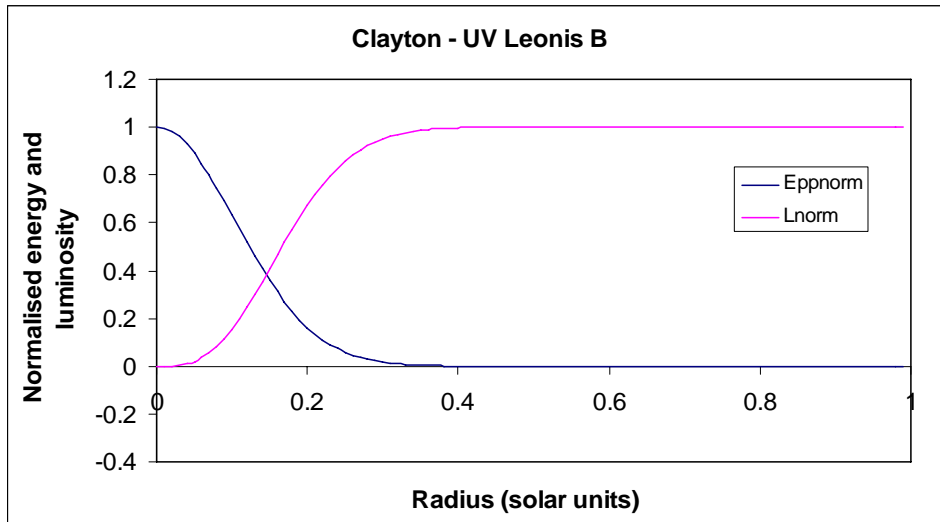


Figure 10. Normalised energy and luminosity from the Clayton model

5. Summary

Figure 7 shows a brief summary some of the main results from both models and the initial calculations.

	P_c (Pa)	$\langle P \rangle$ (Pa)	T_c (K)	$\langle T \rangle$ (K)	E_{pp} (Wm^{-3})	L (W)
Calculation	$>3.79 \times 10^{13}$	5.5×10^{14}	9.37×10^6	3.12×10^6	298	3.9×10^{26}
Polytrope n=1.5	5.6×10^{14}	2.27×10^{14}	6.4×10^6	3.83×10^6	2.11	7.7×10^{24}
Polytrope n=3	4.3×10^{14}	1.9×10^{14}	5.77×10^6	4.38×10^6	0.99	5.0×10^{24}
Clayton	4.9×10^{15}	9.11×10^{14}	1.07×10^7	4.18×10^6	454	2.52×10^{26}

Figure 11. Summary of key results.

From this it is evident that both models show a good representation of the expected attributes and trends of this star. While the Polytrope model was able to produce solutions for a whole range of fitting parameters, it's calculations of the luminosity were a little underestimated, and the fact that there are three different variables to adjust but only two constraints, makes it difficult to narrow down which set of parameters are correct. However, the Clayton model uses only two parameters which depend on each other to fulfill the constraints. This method also produced a set of results which compare well to the estimates based on first principle physics, in particular a better result for the luminosity.

6. Discussion of possible end points of star

By looking at the position of UV Leonis B on the Hertzsprung-Russell diagram it is possible to estimate its lifetime on the main sequence. Like the Sun, it is an average mass and temperature star, finding itself in the middle of the main sequence, with a lifetime of approximately 10^{10} years. After this time the core hydrogen burning will cease, and the star will start its journey to its final fate!

For a low mass star such as UV Leonis B, when the hydrogen burning stops its core shrinks, heating the surrounding hydrogen. Subsequently this triggers hydrogen burning in the shell, and the production of more energy, which causes the stars outer layers to expand and cool, i.e. the luminosity increases and surface temperature drops. The star consequently migrates to the red giant branch of the H-R diagram.

The shrinking and heating of the core continues until helium burning begins. This second stage occurs quite suddenly, with the surrounding hydrogen-burning shell still providing most of the red giant's luminosity. The burning of helium produces carbon and oxygen nuclei and causes the core to expand again, which cools it down. This leads to a slower rate of energy release from the shell, hence a reduced luminosity. The stars outer layers contract and heat, increasing the surface temperature.

When all the helium has been converted to C and O, after approximately 10^8 years the core again contracts until it is stopped by the degenerate electron pressure. Helium burning will begin in the shell, and a second red giant phase is entered, with an even greater luminosity. The outer layers expand once again, ending its evolution by gently expelling its outer layers into space, which form a glowing cloud called a planetary nebula. The burned out core remaining at the centre is a white dwarf, and will continue to cool until it eventually fades away.

7. References

1. Physics of Stars handouts – Ben Murdin
2. An Introduction to Modern Astrophysics – Bradley W. Carroll & Dale A. Ostlie
3. The Physics of Stars – A.C. Phillips
4. Universe – Roger A. Freedman & William J. Kaufmann III

8. Appendix A – Constants and solar values used in calculations

Symbol	Value
M sun	$1.99 \times 10^{30} \text{ kg}$
R sun	$6.96 \times 10^8 \text{ m}$
L sun	$3.83 \times 10^{26} \text{ W}$
$\langle \rho \rangle$ sun	$1.39 \times 10^3 \text{ kg m}^{-3}$
ρ_c sun	$3.82 \times 10^5 \text{ kg m}^{-3}$
$\langle P \rangle$ sun	$1.33 \times 10^{14} \text{ Pa}$
$\langle T \rangle$ sun	$1.16 \times 10^7 \text{ K}$
σ	$5.67 \times 10^{-8} \text{ W m}^{-2} \text{ K}^{-2}$
G	$6.7 \times 10^{-11} \text{ m}^3 \text{ kg}^{-1} \text{ s}^{-2}$
k_B	$1.38 \times 10^{-23} \text{ J K}^{-1}$
a	$7.57 \times 10^{-16} \text{ J K}^{-4}$